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Number Theory

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*An Introduction to the Theory of Numbers*

What Is Number Theory?

Number theory is the study of the most natural of the numbers, those of which we first learned, those of which all other mathematics have been built upon. Number theory is the study of the set of positive whole numbers, most often denoted the Natural Numbers of 1,2,3,4,5,6,7,…,. These numbers are connected and related through a variety of ways. The set of the naturals is sometimes thinned to numbers who share even more in common with one another. Some examples are the odds, the squares, the perfects, the triangular, the Fibonaccis, and the primes. Number theory is about numbers, and the patterns that relate them.

Number theory is subset of pure mathematics. Sometimes denote “The Queen,” it is incredibly fundamental to the workings of mathematics as a whole. The field has a rich history by way of being so close to the core, and from Pythagoras to Diophantus, Euler to Gauss, it has been touched by many a great mathematician all throughout history.

There are several notable subdivisions of the field. Analytical number theory is the study of the integers or the naturals by means of real and complex analysis. For example, sieve theory, a way of created thin or sifted sets of integers, is one of the main techniques mathematicians, even today, solve great theoretical questions. Algebraic number theory is the study of the algebraic numbers, or any complex number that is a solution to some polynomial equation denoted with f(x) with rational coefficients. Diophantine geometry is to determine when a Diophantine equation has a solution and to determine those solutions. All of these fields are united among the common interest of the naturals.

It is often thought that number theory is untouched by application, that it is purely mathematics for mathematics sake. In the modern era, there are many applications for modern number theory that range from computation to chemistry. For example, it is instrumental to the construction of codes and lattice packing’s in hash functions, performing fast arithmetic operations, congruence cryptography, and the periodicity of matter among other things.

Number Theory is a branch of mathematics that ranges from simple arithmetic done by everyone all over the world, to several hundred page proofs understood by just a few. It is truly a deep way of understanding of mathematics and therefore the world and houses great truths about the nature of numbers.

1.0) The Basic Properties of Division

*Definition* Division:

Suppose all variables are members of the naturals:

Theorem 0.1:

Q.E.D

Theorem 1.0.2:

Q.E.D

Theorem 1.0.4

Q.E.D

Theorem 1.0.5a

Q.E.D

Theorem 1.0.5b

Q.E.D

1.1) Triangular Numbers

Definition 1.1, Triangular Number: , or the triangular number, can be written as

. Or in other words:

Lemma 1.1.1) The Symmetry of Triangles

Q.E.D

Theorem 1.1) The Triangular Number Theorem:

Q.E.D

Generating such can be shown as follows in Java:

import acm.program.\*;

public class AllTriangularNumbers extends ConsoleProgram

{

public void run()

{

long i = 0;

long startTime = System.currentTimeMillis();

while(false||(System.currentTimeMillis()-startTime)<1000) {

print((i\*i + i)/2 + ", ");

i++;

}

}

}

Triangular Squares are numbers that are both triangular and perfect square, for example, 36. A trivial way of generating such numbers can be implimented in Java in the following way:

import acm.program.\*;

public class TriangleSquares extends ConsoleProgram

{

public void run()

{

long i = 0;

while(true) {

if(isTriangleNumber(i) && isPerfectSquare(i))

print(i + ", ");

i++;

}

}

public boolean isPerfectSquare(long i) {

long closestRoot = (long) Math.sqrt(i);

return i == closestRoot \* closestRoot;

}

public boolean isTriangleNumber(long i) {

double r1, r2;

r1 = ((-1 + Math.sqrt(1 - 4\*-2\*i)) / (2));

r2 = ((-1 - Math.sqrt(1 - 4\*-2\*i)) / (2));

return (r1 == (int)r1 && r1 > 0 || r2 == (int)r2 && r2 > 0);

}

}

This topic is actually immensely deep. This sieve set is quite rare and fascinating. There will be more on triangular numbers later in this study.

For example, the following program should be noted to show that the ratio of a triangular number to the perfect square is .

import acm.program.\*;

public class RatioOfTriangularNumbers extends ConsoleProgram

{

public void run()

{

long i = 0;

while(true) {

if(isTriangleNumber(i) && isPerfectSquare(i))

println("number: " + i + " Square: " + Math.sqrt(i) + " Triangular Number: " + triangularNumber(i) + " Ratio: " + triangularNumber(i)/Math.sqrt(i));

i++;

}

}

public long triangularNumber(long i) {

return (long)((-1 + Math.sqrt(1 - 4\*-2\*i)) / (2));

}

public boolean isPerfectSquare(long i) {

long closestRoot = (long) Math.sqrt(i);

return i == closestRoot \* closestRoot;

}

public boolean isTriangleNumber(long i) {

double r1, r2;

r1 = ((-1 + Math.sqrt(1 - 4\*-2\*i)) / (2));

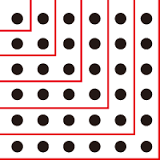
return (r1 == (int)r1 && r1 > 0);

}

}

1.2) The Sum of Odd Numbers

The Odd Number Theorem is as follows:



The sum of Odd Numbers can be simply geometrically represented by the diagram to the left. This can be algebraically represented by the following equation:

A converse also does indeed exist,

This is relatively intuitive and quite an attainable and fascinating principle of theory. I would like to note that this may be a good pattern to share in an attainable talk at the end of the semester.

Twin Primes Java Program

import java.util.Arrays;

import acm.program.\*;

public class twinPrimes extends ConsoleProgram

{

public void run()

{

printSieve(100000,true);

}

public void printSieve(int upThrough, boolean wantTwins) {

boolean[] primes;

if(wantTwins) primes= fillSieveOfTwinPrimes(upThrough);

else primes= fillSieveOfPrimes(upThrough);

for(int i=0;i<primes.length;i++)

if(primes[i])

print(i +", ");

println("done");

}

public boolean[] fillSieveOfPrimes(int upThrough) {

boolean[] a = new boolean[upThrough];

Arrays.fill(a,true);

a[0]=a[1]=false;

for (int i=2;i<a.length;i++)

if(a[i])

for (int j=2;i\*j<a.length;j++)

a[i\*j]=false;

return a;

}

public boolean[] fillSieveOfTwinPrimes(int upThrough) {

boolean[] b = fillSieveOfPrimes(upThrough);

for(int i=0;i<b.length;i++)

if(b[i] && !b[i+2] &!b[i-2])

b[i]=false;

return b;

}

boolean isPrime(int n) {

if (n == 2) return true;

if (n%2==0) return false;

for(int i=3;i\*i<=n;i+=2) {

if(n%i==0)

return false;

}

return true;

}

}

On Primes of the form N2+a

This problem is that of the subclass of the Bunyakovsky conjecture. This conjecture states that any polynomial, denoted by f(x) in one variable with a positive degree and integer coefficients has infinitely many solutions that are prime numbers. The conjecture was first stated in 1857 by the famous Russian mathematician Viktor Bunyakovsky and he outlined two requisites:

1. The coefficient of the leading term is positive
2. The polynomial is irreducible over the integers

For the purposes of being thorough, all infinite series will be references by their OEIS numbers that can be referenced elsewhere.

*The case of f(x) = x2+1:*

This problem, still unsolved, has the available sequence *AA002496* that is shown below:

n f(n)

|  |  |
| --- | --- |
| 1 | 2 |
| 2 | 5 |
| 4 | 17 |
| 6 | 37 |
| 10 | 101 |
| 14 | 197 |
| 16 | 256 |
| 20 | 401 |

The conjecture that this function, denoted *f(n),* should be prime infinitely often was raised by Euler, is the firth Hardy-Littlewood conjecture, and the fourth Landau’s problem, and still remains unsolved.

This problem (1.4) addressed in “A Friendly Introduction To Modern Number Theory” (that will be denoted FIMT) has several more extensions. My answers are as follows:

There are not infinitely many primes of the form N2-1 as that polynomial is a perfect square of the form (N+1)(N-1). The definition of a prime is that it has only two factors, itself and 1. Therefore, to be a prime expressed in the form N2-1, the factor (N-1) must be 1, and if it is not, then that number is not prime. Thus, 3 is the only prime that follows this pattern (N = 2) and there are not infinitely many primes of the form N2-1. This goes by extension for any perfect square of the form N2-a, meaning the only prime that satisfies those conditions are when N – sqrt(a) = 1. For example, N2-4 only has the prime number 5 satisfy the function with the input of 3. Thus, any polynomial that is reducible over the integers will not satisfy this conjecture. However, for the other cases mentioned in 1.4, I cannot prove, but only suspect that the conjecture holds as the function would be a counter example of the Bunyakovsky conjecture if that were the case.